

# L-PACKETS AND DEPTH FOR $\mathrm{SL}_2(K)$ WITH $K$ A LOCAL FUNCTION FIELD OF CHARACTERISTIC 2

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ABSTRACT. Let  $\mathcal{G} = \mathrm{SL}_2(K)$  with  $K$  a local function field of characteristic 2. We review Artin-Schreier theory for the field  $K$ , and show that this leads to a parametrization of certain  $L$ -packets in the smooth dual of  $\mathcal{G}$ . We relate this to a recent geometric conjecture. The  $L$ -packets in the principal series are parametrized by quadratic extensions, and the supercuspidal  $L$ -packets of cardinality 4 are parametrised by biquadratic extensions. Each supercuspidal packet of cardinality 4 is accompanied by a singleton packet for  $\mathrm{SL}_1(D)$ . We compute the depths of the irreducible constituents of all these  $L$ -packets for  $\mathrm{SL}_2(K)$  and its inner form  $\mathrm{SL}_1(D)$ .

## 1. INTRODUCTION

The special linear group  $\mathrm{SL}_2$  has been a mainstay of representation theory for at least 45 years, see [GGPS]. In that book, the authors show how the unitary irreducible representations of  $\mathrm{SL}_2(\mathbb{R})$  and  $\mathrm{SL}_2(\mathbb{Q}_p)$  can be woven together in the context of automorphic forms. This comes about in the following way. The classical notion of a cusp form  $f$  in the upper half plane leads first to the concept of a cusp form on the adèle group of  $\mathrm{GL}_2$  over  $\mathbb{Q}$ , and thence to the idea of an automorphic cuspidal representation  $\pi_f$  of the adèle group of  $\mathrm{GL}_2$ . We recall that the adèle group of  $\mathrm{GL}_2$  is the restricted product of the local groups  $\mathrm{GL}_2(\mathbb{Q}_p)$  where  $p$  is a place of  $\mathbb{Q}$ . If  $p$  is infinite then  $\mathbb{Q}_p$  is the real field  $\mathbb{R}$ ; if  $p$  is finite then  $\mathbb{Q}_p$  is the  $p$ -adic field. The unitary representation  $\pi_f$  may be expressed as  $\otimes \pi_p$  with one local representation for each local group  $\mathrm{GL}_2(\mathbb{Q}_p)$ . It is this way that the unitary representation theory of groups such as  $\mathrm{GL}_2(\mathbb{Q}_p)$  enters into the modern theory of automorphic forms.

Let  $X$  be a smooth projective curve over  $\mathbb{F}_q$ . Denote by  $F$  the field  $\mathbb{F}_q(X)$  of rational functions on  $X$ . For any closed point  $x$  of  $X$  we denote by  $F_x$  the completion of  $F$  at  $x$  and by  $\mathfrak{o}_x$  its ring of integers. If we choose a local coordinate  $t_x$  at  $x$  (i.e., a rational function on  $X$  which vanishes at  $x$  to order one), then we obtain isomorphisms  $F_x \simeq \mathbb{F}_{q_x}((t_x))$  and  $\mathfrak{o}_x \simeq \mathbb{F}_{q_x}[[t_x]]$ , where  $\mathbb{F}_{q_x}$  is the residue field of  $x$ ; in general, it is a finite extension of  $\mathbb{F}_q$  containing  $q_x = q^{\deg(x)}$  elements. Thus, we now have a *local function field* attached to each point of  $X$ .

With all this in the background, it seems natural to us to study the representation theory of  $\mathrm{SL}_2(K)$  with  $K$  a local function field. The case when  $K$  has characteristic 2 has many special features – and we focus on this case in this article. A local function field  $K$  of characteristic 2 is of the form  $K = \mathbb{F}_q((t))$ , the field of Laurent series with coefficients in  $\mathbb{F}_q$ , with  $q = 2^f$ . This example is particularly interesting because there are countably many quadratic extensions of  $\mathbb{F}_q((t))$ .

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Artin-Schreier theory is a branch of Galois theory, and more specifically is a positive characteristic analogue of Kummer theory, for Galois extensions of degree equal to the characteristic  $p$ . Artin and Schreier (1927) introduced Artin-Schreier theory for extensions of prime degree  $p$ , and Witt (1936) generalized it to extensions of prime power degree  $p^n$ . If  $K$  is a field of characteristic  $p$ , a prime number, any polynomial of the form

$$X^p - X + \alpha$$

for  $\alpha \in K$ , is called an Artin-Schreier polynomial. When  $\alpha$  does not lie in the subset  $\{y \in K \mid y = x^p - x \text{ for } x \in K\}$ , this polynomial is irreducible in  $K[X]$ , and its splitting field over  $K$  is a cyclic extension of  $K$  of degree  $p$ . This follows since for any root  $\beta$ , the numbers  $\beta + i$ , for  $1 \leq i \leq p$ , form all the root – by Fermat’s little theorem – so the splitting field is  $K(\beta)$ . Conversely, any Galois extension of  $K$  of degree  $p$  equal to the characteristic of  $K$  is the splitting field of an Artin-Schreier polynomial. This can be proved using additive counterparts of the methods involved in Kummer theory, such as Hilbert’s theorem 90 and additive Galois cohomology. These extensions are called Artin-Schreier extensions.

For the moment, let  $F$  be a local nonarchimedean field with odd residual characteristic. The  $L$ -packets for  $\mathrm{SL}_2(F)$  are classified in the paper [LR] by Lansky-Rhaguram. They comprise: the principal series  $L$ -packets  $\xi_E = \{\pi_E^1, \pi_E^2\}$  where  $E/F$  is a quadratic extension; the unramified supercuspidal  $L$ -packet of cardinality 4; and the supercuspidal  $L$ -packets of cardinality 2.

We now revert to the case of a local function field  $K$  of characteristic 2. We consider  $\mathrm{SL}_2(K)$ . Drawing on the accounts in [Da, Th1, Th2], we review Artin-Schreier theory, adapted to the local function field  $K$ , with special emphasis on the quadratic extensions of  $K$ .

The  $L$ -packets in the principal series of  $\mathrm{SL}_2(K)$  are parametrized by quadratic extensions, and the supercuspidal  $L$ -packets of cardinality 4 are parametrised by bi-quadratic extensions  $L/K$ . There are countably many such supercuspidal  $L$ -packets. In this article, we do not consider supercuspidal  $L$ -packets of cardinality 2.

The concept of *depth* can be traced back to the concept of *level* of a character. Let  $\chi$  be a non-trivial character of  $K^\times$ . The level of  $\chi$  is the least integer  $n \geq 0$  such that  $\chi$  is trivial on the higher unit group  $U_K^{n+1}$ , see [BH, p.12]. The depth of a Langlands parameter  $\phi$  is defined as follows. Let  $r$  be a real number,  $r \geq 0$ , let  $\mathrm{Gal}(K_s/K)^r$  be the  $r$ -th ramification subgroup of the absolute Galois group of  $K$ . Then the depth of  $\phi$  is the smallest number  $d(\phi) \geq 0$  such that  $\phi$  is trivial on  $\mathrm{Gal}(K_s/K)^r$  for all  $r > d(\phi)$ .

The *depth*  $d(\pi)$  of an irreducible  $\mathcal{G}$ -representation  $\pi$  was defined by Moy and Prasad [MoPr1, MoPr2] in terms of filtrations  $P_{x,r}$  ( $r \in \mathbb{R}_{\geq 0}$ ) of the parahoric subgroups  $P_x \subset \mathcal{G}$ .

Let  $\mathcal{G} = \mathrm{SL}_2(K)$ . Let  $\mathbf{Irr}(\mathcal{G})$  denote the smooth dual of  $\mathcal{G}$ . Thanks to a recent article [ABPS1], we have, for every Langlands parameter  $\phi \in \Phi(\mathcal{G})$  with  $L$ -packet  $\Pi_\phi(\mathcal{G}) \subset \mathbf{Irr}(\mathcal{G})$

$$(1) \quad d(\phi) = d(\pi) \quad \text{for all } \pi \in \Pi_\phi(\mathcal{G}).$$

The equation (1) is a big help in the computation of the depth  $d(\pi)$ . To each biquadratic extension  $L/K$ , there is attached a Langlands parameter  $\phi = \phi_{L/K}$ , and an  $L$ -packet  $\Pi_\phi$  of cardinality 4. The depth of the parameter  $\phi_{L/K}$  depends on the extension  $L/K$ . More precisely, the numbers  $d(\phi)$  depend on the breaks in the

upper ramification filtration of the Galois group

$$\mathrm{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

For certain extensions  $L/K$  the allowed depths can be any odd number  $1, 3, 5, 7, \dots$ . For the other extensions  $L/K$ , the allowed depths are  $3, 5, 7, 9, \dots$ . Accordingly, the depth of each irreducible supercuspidal representation  $\pi$  in the packet  $\Pi_\phi$  is given by the formula

$$(2) \quad d(\pi) = 2n + 1$$

where  $n = 0, 1, 2, 3, \dots$  or  $1, 2, 3, 4, \dots$  depending on  $L/K$ . Let  $D$  be a central division algebra of dimension 4 over  $K$ . The parameter  $\phi$  is relevant for the inner form  $\mathrm{SL}_1(D)$ , which admits singleton  $L$ -packets, and the depths are given by the formula (2).

This contrasts with the case of  $\mathrm{SL}_2(\mathbb{Q}_p)$  with  $p > 2$ . Here there is a unique bi-quadratic extension  $L/K$ , and a unique tamely ramified parameter  $\phi : \mathrm{Gal}(L/K) \rightarrow \mathrm{SO}_3(\mathbb{R})$  of depth zero.

We move on to consider the geometric conjecture in [ABPS]. Let  $\mathfrak{B}(\mathcal{G})$  denote the Bernstein spectrum of  $\mathcal{G}$ , let  $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ , and let  $T^\mathfrak{s}, W^\mathfrak{s}$  denote the complex torus, finite group, attached by Bernstein to  $\mathfrak{s}$ . For more details at this point, we refer the reader to [R]. The Bernstein decomposition provides us, inter alia, with the following data: a canonical disjoint union

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^\mathfrak{s}$$

and, for each  $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ , a finite-to-one surjective map

$$\mathbf{Irr}(\mathcal{G})^\mathfrak{s} \rightarrow T^\mathfrak{s}/W^\mathfrak{s}$$

onto the quotient variety  $T^\mathfrak{s}/W^\mathfrak{s}$ . The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a *bijection*

$$(3) \quad \mathbf{Irr}(\mathcal{G})^\mathfrak{s} \simeq T^\mathfrak{s} // W^\mathfrak{s}$$

where  $T^\mathfrak{s} // W^\mathfrak{s}$  is the *extended quotient* of the torus  $T^\mathfrak{s}$  by the finite group  $W^\mathfrak{s}$ . If the action of  $W^\mathfrak{s}$  on  $T^\mathfrak{s}$  is free, then the extended quotient is equal to the ordinary quotient  $T^\mathfrak{s}/W^\mathfrak{s}$ . If the action is not free, then the extended quotient is a finite disjoint union of quotient varieties, one of which is the ordinary quotient. The bijection (3) is subject to certain constraints, itemised in [ABPS].

In the case of  $\mathrm{SL}_2$ , the torus  $T^\mathfrak{s}$  is of dimension 1, and the finite group  $W^\mathfrak{s}$  is either 1 or  $\mathbb{Z}/2\mathbb{Z}$ . So, in this context, the content of the conjecture is rather modest: but a proof is required, and such a proof is duly given in §7.

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## 2. ARTIN-SCHREIER THEORY

Let  $K$  be a local field with positive characteristic  $p$ , containing the  $n$ -th roots of unity  $\zeta_n$ . The cyclic extensions of  $K$  whose degree  $n$  is coprime with  $p$  are described by Kummer theory. It is well known that any cyclic extension  $L/K$  of degree  $n$ ,  $(n, p) = 1$ , is generated by a root  $\alpha$  of an irreducible polynomial  $x^n - a \in K[x]$ . We fix an algebraic closure  $\overline{K}$  of  $K$  and a separable closure  $K^s$  of  $K$  in  $\overline{K}$ . If  $\alpha \in K^s$

is a root of  $x^n - a$  then  $K(\alpha)/K$  is a cyclic extension of degree  $n$  and is called a Kummer extension of  $K$ .

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by  $ch(K) = p$ . It is therefore an analogue of Kummer theory, where the role of the polynomial  $x^n - a$  is played by  $x^n - x - a$ . Essentially, every cyclic extension of  $K$  with degree  $p = ch(K)$  is generated by a root  $\alpha$  of  $x^p - x - a \in K[x]$ .

Let  $\wp$  denote the Artin-Schreier endomorphism of the additive group  $K^s$  [Ne]:

$$\wp : K^s \rightarrow K^s, \quad x \mapsto x^p - x.$$

Given  $a \in K$  denote by  $K(\wp^{-1}(a))$  the extension  $K(\alpha)$ , where  $\wp(\alpha) = a$  and  $\alpha \in K^s$ . We have the following characterization of finite cyclic Artin-Schreier extensions of degree  $p$ :

**Theorem 2.1.** (i) *Given  $a \in K$ , either  $\wp(x) - a \in K[x]$  has one root in  $K$  in which case it has all the  $p$  roots are in  $K$ , or is irreducible.*  
(ii) *If  $\wp(x) - a \in K[x]$  is irreducible then  $K(\wp^{-1}(a))/K$  is a cyclic extension of degree  $p$ , with  $\wp^{-1}(a) \subset K^s$ .*  
(iii) *If  $L/K$  be a finite cyclic extension of degree  $p$ , then  $L = K(\wp^{-1}(a))$ , for some  $a \in K$ .*

(See [Th1, p.34] for more details.)

We fix now some notation.  $K$  is a local field with characteristic  $p > 1$  with finite residue field  $k$ . The field of constants  $k = \mathbb{F}_q$  is a finite extension of  $\mathbb{F}_p$ , with degree  $[k : \mathbb{F}_p] = f$  and  $q = p^f$ .

Let  $\mathfrak{o}$  be the ring of integers in  $K$  and denote by  $\mathfrak{p} \subset \mathfrak{o}$  the (unique) maximal ideal of  $\mathfrak{o}$ . This ideal is principal and any generator of  $\mathfrak{p}$  is called a uniformizer. A choice of uniformizer  $\varpi \in \mathfrak{o}$  determines isomorphisms  $K \cong \mathbb{F}_q((\varpi))$ ,  $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$  and  $\mathfrak{p} = \varpi\mathfrak{o} \cong \varpi\mathbb{F}_q[[\varpi]]$ .

A normalized valuation on  $K$  will be denoted by  $\nu$ , so that  $\nu(\varpi) = 1$  and  $\nu(K) = \mathbb{Z}$ . The group of units is denoted by  $\mathfrak{o}^\times$ .

**2.1. The Artin-Schreier symbol.** Let  $L/K$  be a finite Galois extension. Let  $N_{L/K}$  be the norm map and denote by  $\text{Gal}(L/K)^{ab}$  the abelianization of  $\text{Gal}(L/K)$ . The reciprocity map is a group isomorphism

$$(4) \quad K^\times / N_{L/K} L^\times \xrightarrow{\cong} \text{Gal}(L/K)^{ab}.$$

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism  $K^\times \rightarrow K^\times / N_{L/K} L^\times$

$$(5) \quad b \in K^\times \mapsto (b, L/K) \in \text{Gal}(L/K)^{ab}.$$

From the Artin symbol we obtain a pairing

$$(6) \quad K \times K^\times \longrightarrow \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha,$$

where  $\wp(\alpha) = a$ ,  $\alpha \in K^s$  and  $L = K(\alpha)$ .

**Definition 2.2.** *Given  $a \in K$  and  $b \in K^\times$ , the Artin-Schreier symbol is defined by*

$$[a, b] = (b, L/K)(\alpha) - \alpha.$$

The Artin-Schreier symbol is a bilinear map satisfying the following properties, see [Ne, p.341]:

- (7)  $[a_1 + a_2, b] = [a_1, b] + [a_2, b];$
- (8)  $[a, b_1 b_2] = [a, b_1] + [a, b_2];$
- (9)  $[a, b] = 0, \forall a \in K \Leftrightarrow b \in N_{L/K} L^\times, L = K(\alpha) \text{ and } \wp(\alpha) = a;$
- (10)  $[a, b] = 0, \forall b \in K^\times \Leftrightarrow a \in \wp(K).$

**2.2. The groups  $K/\wp(K)$  and  $K^\times/K^{\times p}$ .** In this section we recall some properties of the groups  $K/\wp(K)$  and  $K^\times/K^{\times p}$  and use them to redefine the pairing (6). Dalawat [Da2, Da] interprets  $K/\wp(K)$  and  $K^\times/K^{\times p}$  as  $\mathbb{F}_p$ -spaces. This interpretation will be particularly useful in §4.

Consider the additive group  $K$ . By [Da, Proposition 11], the  $\mathbb{F}_p$ -space  $K/\wp(K)$  is countably infinite. Hence,  $K/\wp(K)$  is infinite as a group.

**Proposition 2.3.**  *$K/\wp(K)$  is a discrete abelian torsion group.*

*Proof.* The ring of integers decomposes as a (direct) sum

$$\mathfrak{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \wp(\mathfrak{p}).$$

The restriction  $\wp : \mathfrak{p} \rightarrow \mathfrak{p}$  is an isomorphism, see [Da, Lemma 8]. Hence,

$$\wp(\mathfrak{o}) = \wp(\mathbb{F}_q) + \mathfrak{p}$$

and  $\mathfrak{p} \subset \wp(K)$ . It follows that  $\wp(K)$  is an open subgroup of  $K$  and  $K/\wp(K)$  is discrete. Since  $\wp(K)$  is annihilated by  $p$ ,  $K/\wp(K)$  is a torsion group.  $\square$

Now we concentrate on the multiplicative group  $K^\times$ . For any  $n > 0$ , let  $U_n$  be the kernel of the reduction map from  $\mathfrak{o}^\times$  to  $(\mathfrak{o}/\mathfrak{p}^n)^\times$ . In particular,  $U_1 = \ker(\mathfrak{o}^\times \rightarrow k^\times)$ . The  $U_n$  are  $\mathbb{Z}_p$ -modules, because they are commutative pro- $p$ -groups. By [Da2, Proposition 20], the  $\mathbb{Z}_p$ -module  $U_1$  is not finitely generated. As a consequence,  $K^\times/K^{\times p}$  is infinite, see [Da2, Corollary 21]. The next result gives a characterization of the topological group  $K^\times/K^{\times p}$ .

**Proposition 2.4.**  *$K^\times/K^{\times p}$  is a profinite abelian  $p$ -torsion group.*

*Proof.* There is a canonical isomorphism  $K^\times \cong \mathbb{Z} \times \mathfrak{o}^\times$ . The group of units is a direct product  $\mathfrak{o}^\times \cong \mathbb{F}_q^\times \times U_1$ , with  $q = p^f$ . By [Iw, p.25], the group  $U_1$  is a direct product of countable many copies of the ring of  $p$ -adic integers

$$U_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \dots = \prod_{\mathbb{N}} \mathbb{Z}_p.$$

Give  $\mathbb{Z}$  the discrete topology and  $\mathbb{Z}_p$  the  $p$ -adic topology. Then, for the product topology,  $K^\times = \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p$  is a topological group, locally compact, Hausdorff and totally disconnected.

Now,  $K^{\times p}$  decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times \dots$$

$$= p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p.$$

Note that  $p\mathbb{Z}/(q-1)\mathbb{Z} = \mathbb{Z}/(q-1)\mathbb{Z}$ , since  $p$  and  $q-1$  are coprime. Denote by  $z = \prod_n z_n$  an element of  $\prod_{\mathbb{N}} \mathbb{Z}_p$ , where  $z_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p$ , for every  $n$ .

The map

$$\varphi : \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p \rightarrow \mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$$

defined by

$$(x, y, z) \mapsto (x \bmod p, \prod_n pr_0(z_n))$$

where  $pr_0(z_n) = a_{0,n}$  is the projection, is clearly a group homomorphism.

Now,  $\mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$  is a topological group for the product topology, where each component  $\mathbb{Z}/p\mathbb{Z}$  has the discrete topology. It is compact, Hausdorff and totally disconnected. Therefore,  $\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$  is a profinite group.

Since

$$\ker \varphi = p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p,$$

it follows that there is an isomorphism of topological groups

$$K^{\times}/K^{\times p} \cong \prod_{\mathbb{N}_0} \mathbb{Z}/p\mathbb{Z},$$

where  $K^{\times}/K^{\times p}$  is given the quotient topology. Therefore,  $K^{\times}/K^{\times p}$  is profinite.  $\square$

From propositions 2.3 and 2.4,  $K/\wp(K)$  is a discrete abelian group and  $K/K^{\times p}$  is an abelian profinite group, both annihilated by  $p = \text{ch}(K)$ . Therefore, Pontryagin duality coincides with  $\text{Hom}(-, \mathbb{Z}/p\mathbb{Z})$  on both of these groups, see [Th2]. The pairing (6) restricts to a pairing

$$(11) \quad [., .] : K/\wp(K) \times K^{\times}/K^{\times p} \rightarrow \mathbb{Z}/p\mathbb{Z}.$$

which we refer from now on to the **Artin-Schreier pairing**. It follows from (9) and (10), that the pairing is nondegenerate (see also [Th2, Proposition 3.1]). The next result shows that the pairing is perfect.

**Proposition 2.5.** *The Artin-Schreier symbol induces isomorphisms of topological groups*

$$K^{\times}/K^{\times p} \xrightarrow{\cong} \text{Hom}(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), bK^{\times p} \mapsto (a + \wp(K) \mapsto [a, b])$$

and

$$K/\wp(K) \xrightarrow{\cong} \text{Hom}(K^{\times}/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), a + \wp(K) \mapsto (bK^{\times p} \mapsto [a, b])$$

*Proof.* The result follows by taking  $n = 1$  in Proposition 5.1 of [Th2], and from the fact that Pontryagin duality for the groups  $K/\wp(K)$  and  $K^{\times}/K^{\times p}$  coincide with  $\text{Hom}(-, \mathbb{Z}/p\mathbb{Z})$  duality. Hence, there is an isomorphism of topological groups between each such group and its bidual.  $\square$

Let  $B$  be a subgroup of the additive group of  $K$  with finite index such that  $\wp(K) \subseteq B \subseteq K$ . The composite of two finite abelian Galois extensions of exponent  $p$  is again a finite abelian Galois extension of exponent  $p$ . Therefore, the composite

$$K_B = K(\wp^{-1}(B)) = \prod_{a \in B} K(\wp^{-1}(a))$$

is a finite abelian Galois extension of exponent  $p$ . On the other hand, if  $L/K$  is a finite abelian Galois extension of exponent  $p$ , then  $L = K_B$  for some subgroup  $\wp(K) \subseteq B \subseteq K$  with finite index.

All such extensions lie in the maximal abelian extension of exponent  $p$ , which we denote by  $K_p = K(\wp^{-1}(K))$ . The extension  $K_p/K$  is infinite and Galois. The corresponding Galois group  $G_p = \text{Gal}(K_p/K)$  is an infinite profinite group and may be identified, under class field theory, with  $K^\times/K^{\times p}$ , see [Th2, Proposition 5.1]. The case  $\text{ch}(K) = 2$  leads to  $G_2 \cong K^\times/K^{\times 2}$  and will play a fundamental role in the sequel.

### 3. QUADRATIC CHARACTERS

From now on we take  $K$  to be a local function field with  $\text{ch}(K) = 2$ . Therefore,  $K$  is of the form  $\mathbb{F}_q((\varpi))$  with  $q = 2^f$ .

When  $K = \mathbb{F}_q((\varpi))$ , we have, according to [Iw, p.25],

$$U_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots = \prod_{\mathbb{N}} \mathbb{Z}_2$$

with countably infinite many copies of  $\mathbb{Z}_2$ , the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of  $K = \mathbb{F}_q((\varpi))$ . By proposition 2.4, there is a bijection between the set of quadratic extensions of  $\mathbb{F}_q((\varpi))$  and the group

$$\mathbb{F}_q((\varpi))^\times / \mathbb{F}_q((\varpi))^{\times 2} \cong \prod_{\mathbb{N}_0} \mathbb{Z}/2\mathbb{Z} = G_2$$

where  $G_2$  is the Galois group of the *maximal abelian extension of exponent 2*. Since  $G_2$  is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension  $K(\alpha)/K$ , with  $\alpha^2 - \alpha = a$ , we associate the Artin-Schreier symbol

$$[a, \cdot) : K^\times / K^{\times 2} \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

Now, let  $\varphi$  denote the isomorphism  $\mathbb{Z}/2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$  with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character

$$(12) \quad \chi_a : K^\times \rightarrow \mathbb{C}^\times, \quad \chi_a = \varphi([a, \cdot))$$

Proposition 2.5 shows that every quadratic character of  $\mathbb{F}_q((\varpi))^\times$  arises in this way.

**Example 3.1.** *The unramified quadratic extension of  $K$  is  $K(\wp^{-1}(\mathfrak{o}))$ , see [Da] proposition 12. According to Dalawat, the group  $K/\wp(K)$  may be regarded as an  $\mathbb{F}_2$ -space and the image of  $\mathfrak{o}$  under the canonical surjection  $K \rightarrow K/\wp(K)$  is an  $\mathbb{F}_2$ -line, i.e., isomorphic to  $\mathbb{F}_2$ . Since  $\wp|_{\mathfrak{p}} : \mathfrak{p} \rightarrow \mathfrak{p}$  is an isomorphism, the image of  $\mathfrak{p}$  in  $K/\wp(K)$  is  $\{0\}$ , see lemma 8 in [Da]. Now, choose any  $a_0 \in \mathfrak{o} \setminus \mathfrak{p}$  such that the image of  $a_0$  in  $\mathfrak{o}/\mathfrak{p}$  has nonzero trace in  $\mathbb{F}_2$ , see [Da, Proposition 9]. The*

quadratic character  $\chi_{a_0} = \varphi([a_0, \cdot])$  associated with  $K(\wp^{-1}(\mathfrak{o}))$  via class field theory is precisely the unramified character  $(n \mapsto (-1)^n)$  from above. Note that any other choice  $b_0 \in \mathfrak{o} \setminus \mathfrak{p}$ , with  $a_0 \neq b_0$ , gives the same unique unramified character, since there is only one nontrivial coset  $a_0 + \wp(K)$  for  $a_0 \in \mathfrak{o} \setminus \mathfrak{p}$ .

Let  $\mathcal{G}$  denote  $\mathrm{SL}_2(K)$ , let  $\mathcal{B}$  be the standard Borel subgroup of  $\mathcal{G}$ , let  $\mathcal{T}$  be the diagonal subgroup of  $\mathcal{G}$ . Let  $\chi$  be a character of  $\mathcal{T}$ . Then,  $\chi$  inflates to a character of  $\mathcal{B}$ . Denote by  $\pi(\chi)$  the (unitarily) induced representation  $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$ . The representation space  $V(\chi)$  of  $\pi(\chi)$  consists of locally constant complex valued functions  $f : \mathcal{G} \rightarrow \mathbb{C}$  such that, for every  $a \in K^\times$ ,  $b \in K$  and  $g \in \mathcal{G}$ , we have

$$f\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g\right) = |a|\chi(a)f(g)$$

The action of  $\mathcal{G}$  on  $V(\chi)$  is by right translation. The representations  $(\pi(\chi), V(\chi))$  are called (unitary) principal series of  $\mathcal{G} = \mathrm{SL}_2(K)$ .

Let  $\chi$  be a quadratic character of  $K^\times$ . The reducibility of the induced representation  $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$  is well known in zero characteristic. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic  $p$ .

**Theorem 3.2.** [Ca, Ca2] *The representation  $\pi(\chi) = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$  is reducible if, and only if,  $\chi$  is either  $|\cdot|^\pm$  or a nontrivial quadratic character of  $K^\times$ .*

For a proof see [Ca, Theorems 1.7, 1.9] and [Ca2, §9].

From now on,  $\chi$  will be a quadratic character. It is a classical result that the unitary principal series for  $\mathrm{GL}_2$  are irreducible. For a representation of  $\mathrm{GL}_2$  parabolically induced by  $1 \otimes \chi$ , Clifford theory tells us that the dimension of the intertwining algebra of its restriction to  $\mathrm{SL}_2$  is 2. This is exactly the induced representation of  $\mathrm{SL}_2$  by  $\chi$ :

$$\mathrm{Ind}_{\tilde{\mathcal{B}}}^{\mathrm{GL}_2(K)}(1 \otimes \chi)|_{\mathrm{SL}(2,K)} \xrightarrow{\sim} \mathrm{Ind}_{\mathcal{B}}^{\mathrm{SL}_2(K)}(\chi)$$

where  $\tilde{\mathcal{B}}$  denotes the standard Borel subgroup of  $\mathrm{GL}_2(K)$ . This leads to reducibility of the induced representation  $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi)$  into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

$$(13) \quad \pi(\chi) = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}(\chi) = \pi(\chi)^+ \oplus \pi(\chi)^-$$

define an  $L$ -packet  $\{\pi(\chi)^+, \pi(\chi)^-\}$  for  $\mathrm{SL}_2$ .

#### 4. BIQUADRATIC EXTENSIONS OF $\mathbb{F}_q((\varpi))$

Quadratic extensions  $L/K$  are obtained by adjoining an  $\mathbb{F}_2$ -line  $D \subset K/\wp(K)$ . Therefore,  $L = K(\wp^{-1}(D)) = K(\alpha)$  where  $D = \mathrm{span}\{a + \wp(K)\}$ , with  $\alpha^2 - \alpha = a$ . In particular, if  $a_0 \in \mathfrak{o} \setminus \mathfrak{p}$  such that the image of  $a_0$  in  $\mathfrak{o}/\mathfrak{p}$  has nonzero trace in  $\mathbb{F}_2$ , the  $\mathbb{F}_2$ -line  $V_0 = \mathrm{span}\{a_0 + \wp(K)\}$  contains all the cosets  $a_i + \wp(K)$  where  $a_i$  is an integer and so  $K(\wp^{-1}(\mathfrak{o})) = K(\wp^{-1}(V_0)) = K(\alpha_0)$  where  $\alpha_0^2 - \alpha_0 = a_0$  gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering  $\mathbb{F}_2$ -planes  $W = \mathrm{span}\{a + \wp(K), b + \wp(K)\} \subset K/\wp(K)$ . Therefore, if  $a + \wp(K)$  and  $b + \wp(K)$  are  $\mathbb{F}_2$ -linearly independent then  $K(\wp^{-1}(W)) := K(\alpha, \beta)$  is biquadratic, where  $\alpha^2 - \alpha = a$  and  $\beta^2 - \beta = b$ ,  $\alpha, \beta \in K^s$ . Therefore,  $K(\alpha, \beta)/K$  is biquadratic if  $b - a \notin \wp(K)$ .



A biquadratic extension containing the line  $V_0$  is of the form  $K(\alpha_0, \beta)/K$ . There are countably many quadratic extensions  $L_0/K$  containing the unramified quadratic extension. They have ramification index  $e(L_0/K) = 2$ . And there are countably many biquadratic extensions  $L/K$  which do not contain the unramified quadratic extension. They have ramification index  $e(L/K) = 4$ .

So, there is a plentiful supply of biquadratic extensions  $K(\alpha, \beta)/K$ .

**4.1. Ramification.** The space  $K/\wp(K)$  comes with a filtration

$$(14) \quad 0 \subset_1 V_0 \subset_f V_1 = V_2 \subset_f V_3 = V_4 \subset_f \dots \subset K/\wp(K)$$

where  $V_0$  is the image of  $\mathfrak{o}_K$  and  $V_i$  ( $i > 0$ ) is the image of  $\mathfrak{p}^{-i}$  under the canonical surjection  $K \rightarrow K/\wp(K)$ . For  $K = \mathbb{F}_q((\varpi))$  and  $i > 0$ , each inclusion  $V_{2i} \subset_f V_{2i+1}$  is a sub- $\mathbb{F}_2$ -space of codimension  $f$ . The  $\mathbb{F}_2$ -dimension of  $V_n$  is

$$(15) \quad \dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil f,$$

for every  $n \in \mathbb{N}$ , where  $\lceil x \rceil$  is the smallest integer bigger than  $x$ .

Let  $L/K$  denote a Galois extension with Galois group  $G$ . For each  $i \geq -1$  we define the  $i^{\text{th}}$ -ramification subgroup of  $G$  (in the lower numbering) to be:

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{p}_L^{i+1}, \forall x \in \mathfrak{o}_L\}.$$

An integer  $t$  is a *break* for the filtration  $\{G_i\}_{i \geq -1}$  if  $G_t \neq G_{t+1}$ . The study of ramification groups  $\{G_i\}_{i \geq -1}$  is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering  $\{G^i\}_{i \geq -1}$  and defined by the *Hasse-Herbrand function*  $\psi = \psi_{L/K}$ :

$$G^u = G_{\psi(u)}.$$

In particular,  $G^{-1} = G_{-1} = G$  and  $G^0 = G_0$ , since  $\psi(0) = 0$ .

Let  $G_2 = \text{Gal}(K_2/K)$  be the Galois group of the maximal abelian extension of exponent 2,  $K_2 = K(\wp^{-1}(K))$ . Since  $G_2 \cong K^\times/K^{\times 2}$  (proposition 2.4), the pairing  $K^\times/K^{\times 2} \times K/\wp(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  from (11) coincides with the pairing  $G_2 \times K/\wp(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$ .

The profinite group  $G_2$  comes equipped with a ramification filtration  $(G_2^u)_{u \geq -1}$  in the upper numbering, see [Da, p.409]. For  $u \geq 0$ , we have an orthogonal relation [Da, Proposition 17]

$$(16) \quad (G_2^u)^\perp = \overline{\mathfrak{p}^{-\lceil u \rceil + 1}} = V_{\lceil u \rceil - 1}$$

under the pairing  $G_2 \times K/\wp(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$ .

Since the upper filtration is more suitable for quotients, we will compute the upper breaks. By using the Hasse-Herbrand function it is then possible to compute the lower breaks in order to obtain the lower ramification filtration.

According to [Da, Proposition 17], the positive breaks in the filtration  $(G^v)_v$  occur precisely at integers prime to  $p$ . So, for  $ch(K) = 2$ , the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If  $G$  is cyclic of prime order, then there is a unique break for any decreasing filtration  $(G^v)_v$  (see [Da, Proposition 14]). In general, the number of breaks depends on the possible filtration of the Galois group.

Given a plane  $W \subset K/\wp(K)$ , the filtration (14)  $(V_i)_i$  on  $K/\wp(K)$  induces a filtration  $(W_i)_i$  on  $W$ , where  $W_i = W \cap V_i$ . There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

**Case 1 :**  $W$  contains the line  $V_0$ , i.e.  $L_0 = K(\wp^{-1}(W))$  contains the unramified quadratic extension  $K(\wp^{-1}(V_0)) = K(\alpha_0)$  of  $K$ . The extension has residue degree  $f(L_0/K) = 2$  and ramification index  $e(L_0/K) = 2$ . In this case, there is an integer  $t > 0$ , necessarily odd, such that the filtration  $(W_i)_i$  looks like

$$0 \subset_1 W_0 = W_{t-1} \subset_1 W_t = W.$$

By the orthogonality relation (16), the upper ramification filtration on  $G = \text{Gal}(L_0/K)$  looks like

$$\{1\} = \dots = G^{t+1} \subset_1 G^t = \dots = G^0 \subset_1 G^{-1} = G$$

Therefore, the upper ramification breaks occur at  $-1$  and  $t$ .

The number of such  $W$  is equal to the number of planes in  $V_t$  containing the line  $V_0$  but not contained in the subspace  $V_{t-1}$ . This number can be computed and equals the number of biquadratic extensions of  $K$  containing the unramified quadratic extensions and with a pair of upper ramification breaks  $(-1, t)$ ,  $t > 0$  and odd. Here is an example.

**Example 4.1.** *The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks  $(-1, 1)$  is equal to the number of planes in an  $1 + f$ -dimensional  $\mathbb{F}_2$ -space, containing the line  $V_0$ . There are precisely*

$$1 + 2 + 2^2 + \dots + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1$$

*of such biquadratic extensions.*

**Case 2.1 :**  $W$  does not contain the line  $V_0$  and the induced filtration on the plane  $W$  looks like

$$0 = W_{t-1} \subset_2 W_t = W$$

for some integer  $t$ , necessarily odd.

The number of such  $W$  is equal to the number of planes in  $V_t$  whose intersection with  $V_{t-1}$  is  $\{0\}$ . Note that, there are no such planes when  $f = 1$ . So, for  $K = \mathbb{F}_2((\varpi))$ , **case 2.1** does not occur.

Suppose  $f > 1$ . By the orthogonality relation, the upper ramification filtration on  $G = \text{Gal}(L/K)$  looks like

$$\{1\} = \dots = G^{t+1} \subset_2 G^t = \dots = G^{-1} = G$$

Therefore, there is a single upper ramification break occurring at  $t > 0$  and is necessarily odd.

For  $f = 1$  there is no such biquadratic extension. For  $f > 1$ , the number of these biquadratic extensions equals the number of planes  $W$  contained in an  $\mathbb{F}_2$ -space of dimension  $1 + fi$ ,  $t = 2i - 1$ , which are transverse to a given codimension- $f$   $\mathbb{F}_2$ -space.

**Case 2.2 :**  $W$  does not contain the line  $V_0$  and the induced filtration on the plane  $W$  looks like

$$0 = W_{t_1-1} \subset_1 W_{t_1} = W_{t_2-1} \subset_1 W_{t_2} = W$$

for some integers  $t_1$  and  $t_2$ , necessarily odd, with  $0 < t_1 < t_2$ .

The orthogonality relation for this case implies that the upper ramification filtration on  $G = \text{Gal}(L/K)$  looks like

$$\{1\} = \dots = G^{t_2+1} \subset_1 G^{t_2} = \dots = G^{t_1+1} \subset_1 G^{t_1} = \dots = G$$

The upper ramification breaks occur at odd integers  $t_1$  and  $t_2$ .

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks  $(t_1, t_2)$ .

## 5. LANGLANDS PARAMETER

We have the following canonical homomorphism:

$$\mathbf{W}_K \rightarrow \mathbf{W}_K^{ab} \simeq K^\times \rightarrow K^\times / K^{\times 2}.$$

According to §2, we also have

$$K^\times / K^{\times 2} \simeq \prod \mathbb{Z}/2\mathbb{Z}$$

the product over countably many copies of  $\mathbb{Z}/2\mathbb{Z}$ . Using the countable axiom of choice, we choose two copies of  $\mathbb{Z}/2\mathbb{Z}$ . This creates a homomorphism

$$\mathbf{W}_K \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

There are countably many such homomorphisms.

Following [We], denote by  $\alpha, \beta, \gamma$  the images in  $\text{PSL}_2(\mathbb{C})$  of the elements

$$z_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad z_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z_\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

in  $\text{SL}_2(\mathbb{C})$ .

Note that  $z_\alpha, z_\beta, z_\gamma \in \text{SU}_2(\mathbb{C})$  so that

$$\alpha, \beta, \gamma \in \text{PSU}_2(\mathbb{C}) = \text{SO}_3(\mathbb{R}).$$

Denote by  $J$  the group generated by  $\alpha, \beta, \gamma$ :

$$J := \{\epsilon, \alpha, \beta, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The group  $J$  is unique up to conjugacy in  $G = \text{PSL}_2(\mathbb{C})$ .

The pre-image of  $J$  in  $\text{SL}_2(\mathbb{C})$  is the group  $\{\pm 1, \pm z_\alpha, \pm z_\beta, \pm z_\gamma\}$  and is isomorphic to the group  $U_8$  of unit quaternions  $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ .

The centralizer and normalizer of  $J$  are given by

$$C_G(J) = J, \quad N_G(J) = O$$

where  $O \simeq S_4$  the symmetric group on 4 letters. The quotient  $O/J \simeq \text{GL}_2(\mathbb{Z}/2)$  is the full automorphism group of  $J$ .

Each biquadratic extension  $L/K$  determines a Langlands parameter

$$(17) \quad \phi : \text{Gal}(L/K) \rightarrow \text{SO}_3(\mathbb{R}) \subset \text{SO}_3(\mathbb{C})$$

Define

$$(18) \quad S_\phi = C_{\text{PSL}_2(\mathbb{C})}(\text{im } \phi)$$

Then we have  $S_\phi = J$ , since  $C_G(J) = J$ , and whose conjugacy class depends only on  $L$ , since  $O/J = \text{Aut}(J)$ .

Define the new group

$$\mathcal{S}_\phi = C_{\text{SL}_2(\mathbb{C})}(\text{im } \phi)$$

To align with the notation in [ABPS2], replace  $\phi^\sharp$  in [ABPS2] by  $\phi$  in the present article. We have the short exact sequence

$$1 \rightarrow \mathcal{Z}_\phi \rightarrow \mathcal{S}_\phi \rightarrow S_\phi \rightarrow 1$$

with  $\mathcal{Z}_\phi = \mathbb{Z}/2\mathbb{Z}$ .

Let  $D$  be a central division algebra of dimension 4 over  $K$ , and let  $\text{Nrd}$  denote the reduced norm on  $D^\times$ . Define

$$\text{SL}_1(D) = \{x \in D^\times : \text{Nrd}(x) = 1\}.$$

Then  $\text{SL}_1(D)$  is an inner form of  $\text{SL}_2(K)$ . In the local Langlands correspondence [ABPS2] for the inner forms of  $\text{SL}_2$ , the L-parameter  $\phi$  is enhanced by elements  $\rho \in \mathbf{Irr}(\mathcal{S}_\phi)$ . Now the group  $\mathcal{S}_\phi \simeq U_8$  admits four characters  $\rho_1, \rho_2, \rho_3, \rho_4$  and one irreducible representation  $\rho_0$  of degree 2.

The parameter  $\phi$  creates a big packet with five elements, which are allocated to  $\text{SL}_2(K)$  or  $\text{SL}_1(D)$  according to central characters. So  $\phi$  assigns an  $L$ -packet  $\Pi_\phi$  to  $\text{SL}_2(K)$  with 4 elements, and a singleton packet to the inner form  $\text{SL}_1(D)$ . None of these packets contains the Steinberg representation of  $\text{SL}_2(K)$  and so each  $\Pi_\phi$  is a supercuspidal  $L$ -packet with 4 elements.

To be explicit:  $\phi$  assigns to  $\text{SL}_2(K)$  the supercuspidal packet

$$\{\pi(\phi, \rho_1), \pi(\phi, \rho_2), \pi(\phi, \rho_3), \pi(\phi, \rho_4)\}$$

and to  $\text{SL}_1(D)$  the singleton packet

$$\{\pi(\phi, \rho_0)\}$$

and this phenomenon occurs countably many times.

Each supercuspidal packet of four elements is the *JL-transfer* of the singleton packet, in the following sense: the irreducible supercuspidal representation  $\theta$  of  $\text{GL}_2(K)$  which yields the 4-packet upon restriction to  $\text{SL}_2(K)$  is the image in the JL-correspondence of the irreducible smooth representation  $\psi$  of  $\text{GL}_1(D)$  which yields two copies of  $\pi(\phi, \rho_0)$  upon restriction to  $\text{SL}_1(D)$ :

$$\theta = JL(\psi).$$

Each parameter  $\phi : \mathbf{W}_K \rightarrow \text{PGL}_2(\mathbb{C})$  lifts to a Galois representation

$$\phi : \mathbf{W}_K \rightarrow \text{GL}_2(\mathbb{C}).$$

This representation is *triply imprimitive*, as in [We]. Let  $\mathfrak{T}(\phi)$  be the group of characters  $\chi$  of  $\mathbf{W}_K$  such that  $\chi \otimes \phi \simeq \phi$ . Then  $\mathfrak{T}(\phi)$  is non-cyclic of order 4.

## 6. DEPTH

Let  $L/K$  be a biquadratic extension. We fix an algebraic closure  $\overline{K}$  of  $K$  such that  $L \subset \overline{K}$ . From the inclusion  $L \subset \overline{K}$ , there is a natural surjection

$$\pi_{L/K} : \text{Gal}(\overline{K}/K) \rightarrow \text{Gal}(L/K)$$

Let  $K^{ur}$  be the maximal unramified extension of  $K$  in  $\overline{K}$  and let  $K^{ab}$  be the maximal abelian extension of  $K$  in  $\overline{K}$ . We have a commutative diagram, where the horizontal maps are the canonical maps and the vertical maps are the natural projections

$$\begin{array}{ccccccc}
1 & \longrightarrow & I_{\overline{K}/K} & \xrightarrow{\iota_1} & \text{Gal}(\overline{K}/K) & \xrightarrow{p_1} & \text{Gal}(K^{ur}/K) \longrightarrow 1 \\
& & \alpha_1 \downarrow & & \pi_1 \downarrow & & id \downarrow \\
1 & \longrightarrow & I_{K^{ab}/K} & \xrightarrow{\iota_2} & \text{Gal}(K^{ab}/K) & \xrightarrow{p_2} & \text{Gal}(K^{ur}/K) \longrightarrow 1 \\
& & \alpha_2 \downarrow & & \pi_2 \downarrow & & \beta \downarrow \\
1 & \longrightarrow & \mathfrak{I}_{L/K} & \xrightarrow{\iota_3} & \text{Gal}(L/K) & \xrightarrow{p_3} & \text{Gal}(L \cap K^{ur}/K) \longrightarrow 1
\end{array}$$

In the above notation, we have  $\pi_{L/K} = \pi_2 \circ \pi_1$ .

Let

$$(19) \quad \dots \mathfrak{I}^{(2)} \subset \mathfrak{I}^{(1)} \subset \mathfrak{I}^{(0)} \subset G = \text{Gal}(L/K)$$

be the filtration of the relative inertia subgroup  $\mathfrak{I}^{(0)} = \mathfrak{I}_{L/K}$  of  $\text{Gal}(L/K)$ ,  $\mathfrak{I}^{(1)}$  is the wild inertia subgroup, and so on... Note that  $\mathfrak{I}^{(r)}$  is the restriction of the filtration  $G^r$  of  $G = \text{Gal}(L/K)$  to the subgroup  $\mathfrak{I}_{L/K}$ , i.e,  $\mathfrak{I}^{(r)} = \iota_3(G^r)$ .

Let

$$(20) \quad \dots I^{(2)} \subset I^{(1)} \subset I^{(0)} \subset G = \text{Gal}(\overline{K}/K)$$

be the filtration of the absolute inertia subgroup  $I^{(0)} = I_{\overline{K}/K}$  of  $\text{Gal}(\overline{K}/K)$ ,  $I^{(1)}$  is the wild inertia subgroup, and so on...

**Lemma 6.1.** *We have*

$$(\forall r) \quad \pi_{L/K} I^{(r)} = \mathfrak{I}^{(r)}$$

*Proof.* This follows immediately from the above diagram. Here, we identify  $I^{(r)}$  with  $\iota_1(I^{(r)})$  and  $\mathfrak{I}^{(r)}$  with  $\iota_3(\mathfrak{I}^{(r)})$ . □

**Lemma 6.2.** *Let  $L/K$  be a biquadratic extension, let  $\phi$  be the Langlands parameter (17),  $\phi = \alpha \circ \pi_{L/K}$  with  $\alpha : \text{Gal}(L/K) \rightarrow \text{SO}_3(\mathbb{R})$ . Then we have  $d(\phi) = r - 1$  where  $r$  is the least integer for which  $\mathfrak{I}^{(r)} = 1$ .*

*Proof.* The depth of a Langlands parameter  $\phi$  is easy to define. For  $r \in \mathbb{R} \geq 0$  let  $\text{Gal}(F_s/F)^r$  be the  $r$ -th ramification subgroup of the absolute Galois group of  $F$ . Then the depth of  $\phi$  is the smallest number  $d(\phi) \geq 0$  such that  $\phi$  is trivial on  $\text{Gal}(F_s/F)^r$  for all  $r > d(\phi)$ .

Note that  $\alpha$  is *injective*. Therefore

$$\phi(I^{(r)}) = 1 \iff (\alpha \circ \pi_{L/K}) I^{(r)} = 1 \iff \alpha(\mathfrak{I}^{(r)}) = 1 \iff \mathfrak{I}^{(r)} = 1.$$

□

For example, the parameter  $\phi$  has depth zero if it is *tamely ramified*, i.e. the least integer  $r$  for which  $\mathfrak{I}^{(r)} = 1$  is  $r = 1$ . The relative wild inertia group is 1, but the relative inertia group is not 1.

**Case 1:** There are two ramification breaks occurring at  $-1$  and some odd integer  $t > 0$ :

$$\{1\} = \dots = \mathfrak{I}^{(t+1)} \subset \mathfrak{I}^{(t)} = \dots \mathfrak{I}^{(0)} = \mathfrak{I}_{L/K} \subset \text{Gal}(L/K), \quad d(\phi) = t$$

The allowed depths are  $1, 3, 5, 7, \dots$

**Case 2.1:** One single ramification break occurs at some odd integer  $t > 0$ :

$$\{1\} = \dots = \mathfrak{I}^{(t+1)} \subset \mathfrak{I}^{(t)} = \dots = \mathfrak{I}^{(0)} = \mathfrak{I}_{L/K} = \text{Gal}(L/K); \quad d(\varphi) = t$$

The allowed depths are  $1, 3, 5, 7, \dots$

**Case 2.2:** There are two ramification breaks occurring at some odd integers  $t_1 < t_2$

$$\{1\} = \dots = \mathfrak{I}^{(t_2+1)} \subset \mathfrak{I}^{(t_2)} = \dots = \mathfrak{I}^{(t_1+1)} \subset \mathfrak{I}^{(t_1)} = \dots = \mathfrak{I}^{(0)} = \mathfrak{I}_{L/K} = \text{Gal}(L/K); \quad d(\varphi) = t_2$$

The allowed depths are  $3, 5, 7, 9, \dots$

(In the above,  $\mathfrak{I}^{(0)} = \mathfrak{I}_{L/K}$ )

**Theorem 6.3.** *Let  $L/K$  be a biquadratic extension, let  $\phi$  be the Langlands parameter (17). For every  $\pi \in \Pi_\phi(\text{SL}_2(K))$  and  $\pi \in \Pi_\phi(\text{SL}_1(D))$  there is an equality of depths:*

$$d(\pi) = d(\phi).$$

*The depth of each element in the  $L$ -packet  $\Pi_\phi$  is given by the largest break in the ramification of the Galois group  $\text{Gal}(L/K)$ . The allowed depths are  $1, 3, 5, 7, \dots$  except in Case 2.2, when the allowed depths are  $3, 5, 7, \dots$*

*Proof.* This follows from Lemma (6.2), the above computations, and Theorem 3.4 in [ABPS1].  $\square$

This contrasts with the case of  $\text{SL}_2(\mathbb{Q}_p)$  with  $p > 2$ . Here there is a unique biquadratic extension  $L/K$ , and a unique tamely ramified parameter  $\phi : \text{Gal}(L/K) \rightarrow \text{SO}_3(\mathbb{R})$  of depth zero.

**6.1. Quadratic extensions.** Let  $E/K$  be a quadratic extension. There are two kinds: the unramified one  $E_0 = K(\alpha_0)$  and countably many totally (and wildly) ramified  $E = K(\alpha)$ .

**Theorem 6.4.** *For the unramified principal series  $L$ -packet  $\{\pi_E^1, \pi_E^2\}$ , we have*

$$d(\pi_E^1) = d(\pi_E^2) = -1.$$

*For the ramified principal series  $L$ -packet  $\{\pi_E^1, \pi_E^2\}$ , we have*

$$d(\pi_E^1) = d(\pi_E^2) = n$$

*with  $n = 1, 2, 3, 4, \dots$*

*Proof.* Case 1:  $E_0/K$  unramified. Then,  $f(E_0/K) = 2$ . In this case, we have  $G_0 = \{1\}$ , and  $G_0 = G^0 = \mathfrak{I}_{E_0/K}$ . There is only one ramification break at  $t = 0$  and the filtration of  $G = \text{Gal}(E_0/K)$  in the upper numbering is

$$\{1\} = G^0 \subset G^{-1} = G = \mathbb{Z}/2\mathbb{Z}.$$

The filtration on the relative inertia  $\mathfrak{I}^{(t)}$  is

$$\{1\} = \mathfrak{I}_{L_0/K} \subset G = \mathbb{Z}/2\mathbb{Z}$$

with only one break at  $t = 0$ . Negative depth, as expected.

Case 2:  $E/K$  is totally ramified. Then,  $e(E/K) = 2$ , which is divisible by the residue degree, so the extension is wildly ramified. In this case, there is one break

at some  $t \geq 1$ . This is because of wild ramification, since  $G^1 = \{1\}$  if and only if the extension is tamely ramified. The filtration of  $G$  in the upper numbering is

$$\{1\} = G^{t+1} \subset G^t = \dots = G^0 = G = \mathbb{Z}/2\mathbb{Z}$$

The filtration on the relative inertia  $\mathfrak{I}^{(r)}$  is

$$\{1\} = \mathfrak{I}^{(t+1)} \subset \mathfrak{I}^{(t)} = \dots = G = \mathbb{Z}/2\mathbb{Z}$$

with only one break at  $t \geq 1$ . □

## 7. A COMMUTATIVE TRIANGLE

In this section we confirm part of the geometric conjecture in [ABPS] for  $\mathrm{SL}_2(\mathbb{F}_q((\varpi)))$ . We begin by recalling the underlying ideas of the conjecture.

Let  $\mathcal{G}$  be the group of  $K$ -points of a connected reductive group over a nonarchimedean local field  $K$ . The Bernstein decomposition provides us, inner alia, with the following data: a canonical disjoint union

$$\mathbf{Irr}(\mathcal{G}) = \bigsqcup \mathbf{Irr}(\mathcal{G})^{\mathfrak{s}}$$

and, for each  $\mathfrak{s} \in \mathfrak{B}(\mathcal{G})$ , a finite-to-one surjective map

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \rightarrow T^{\mathfrak{s}}/W^{\mathfrak{s}}$$

The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a *bijection*

$$\mathbf{Irr}(\mathcal{G})^{\mathfrak{s}} \simeq (T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$$

where  $(T^{\mathfrak{s}}//W^{\mathfrak{s}})_2$  is the *extended quotient of the second kind* of the torus  $T^{\mathfrak{s}}$  by the finite group  $W^{\mathfrak{s}}$ . This bijection is subject to certain constraints, itemised in [ABPS].

We proceed to define the extended quotient of the second kind. Let  $W$  be a finite group and let  $X$  be a complex affine algebraic variety. Suppose that  $W$  is acting on  $X$  as automorphisms of  $X$ . Define

$$\tilde{X}_2 := \{(x, \tau) : \tau \in \mathbf{Irr}(W_x)\}.$$

Then  $W$  acts on  $\tilde{X}_2$ :

$$\alpha(x, \tau) = (\alpha \cdot x, \alpha_* \tau).$$

**Definition 7.1.** *The extended quotient of the second kind is defined as*

$$(X//W)_2 := \tilde{X}_2/W.$$

Thus the extended quotient of the second kind is the ordinary quotient for the action of  $W$  on  $\tilde{X}_2$ .

We recall that  $(G, T)$  are the complex dual groups of  $(\mathcal{G}, \mathcal{T})$ , so that  $G = \mathrm{PSL}_2(\mathbb{C})$ . Let  $\mathbf{W}_K$  denote the Weil group of  $K$ . If  $\phi$  is an  $L$ -parameter

$$\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \rightarrow G$$

then an *enhanced Langlands parameter* is a pair  $(\phi, \rho)$  where  $\phi$  is a parameter and  $\rho \in \mathbf{Irr}(S_{\phi})$ .

**Theorem 7.2.** *Let  $\mathcal{G} = \mathrm{SL}_2(K)$  with  $K = \mathbb{F}_q((\varpi))$ . Let  $\mathfrak{s} = [\mathcal{T}, \chi]_G$  be a point in the Bernstein spectrum for the principal series of  $\mathcal{G}$ . Let  $\mathbf{Irr}(\mathcal{G})^\mathfrak{s}$  be the corresponding Bernstein component in  $\mathbf{Irr}(\mathcal{G})$ . Then there is a commutative triangle of natural bijections*

$$\begin{array}{ccc} & (T^\mathfrak{s} // W^\mathfrak{s})_2 & \\ \swarrow & & \searrow \\ \mathbf{Irr}(\mathcal{G})^\mathfrak{s} & \xrightarrow{\quad\quad\quad} & \mathfrak{L}(G)^\mathfrak{s} \end{array}$$

where  $\mathfrak{L}(G)^\mathfrak{s}$  denotes the equivalence classes of enhanced parameters attached to  $\mathfrak{s}$ .

*Proof.* We recall that  $T^\mathfrak{s} = \{\psi\chi : \psi \in \Psi(\mathcal{T})\}$  where  $\Psi(\mathcal{T})$  is the group of all unramified quasicharacters of  $\mathcal{T}$ . With  $\lambda \in T^\mathfrak{s}$ , we define the parameter  $\phi(\lambda)$  as follows:

$$\phi(\lambda) : W_K \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PSL}_2(\mathbb{C}), \quad (w\Phi_K^n, Y) \mapsto \begin{pmatrix} \lambda(\varpi)^n & 0 \\ 0 & 1 \end{pmatrix}_*$$

where  $A_*$  is the image in  $\mathrm{PSL}_2(\mathbb{C})$  of  $A \in \mathrm{SL}_2(\mathbb{C})$ ,  $Y \in \mathrm{SL}_2(\mathbb{C})$ ,  $w \in I_K$  the inertia group, and  $\Phi_K$  is a geometric Frobenius. Define, as in §3,

$$\pi(\lambda) := \mathrm{Ind}_B^\mathcal{G}(\lambda).$$

**Case 1.**  $\lambda^2 \neq 1$ . Send the pair  $(\lambda, 1) \in T^\mathfrak{s} // W^\mathfrak{s}$  to  $\pi(\lambda) \in \mathbf{Irr}(\mathcal{G})^\mathfrak{s}$  (via the left slanted arrow) and to  $\phi(\lambda) \in \mathfrak{L}(G)^\mathfrak{s}$  (via the right slanted arrow).

**Case 2.** Let  $\lambda^2 = 1, \lambda \neq 1$ . Let  $\phi = \phi(\lambda)$ . To compute  $S_\phi$ , let  $1, w$  be representatives of the Weyl group  $W = W(G)$ . Then we have

$$C_G(\mathrm{im} \phi) = T \sqcup wT$$

So  $\phi$  is a non-discrete parameter, and we have

$$S_\phi \simeq \mathbb{Z}/2\mathbb{Z}.$$

We have two enhanced parameters, namely  $(\phi, 1)$  and  $(\phi, \epsilon)$  where  $\epsilon$  is the non-trivial character of  $\mathbb{Z}/2\mathbb{Z}$ .

Since  $\lambda^2 = 1$ , there is a point of reducibility. We send

$$(\lambda, 1) \mapsto \pi(\lambda)^+, \quad (\lambda, \epsilon) \mapsto \pi(\lambda)^-$$

via the left slanted arrow, and

$$(\lambda, 1) \mapsto (\phi(\lambda), 1), \quad (\lambda, \epsilon) \mapsto (\phi(\lambda), \epsilon)$$

via the right slanted arrow. Note that this *includes* the case when  $\lambda$  is the unramified quadratic character of  $K^\times$ .

**Case 3.** Let  $\lambda = 1$ . The *principal parameter*

$$\phi_0 : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PSL}(2, \mathbb{C}).$$

is a discrete parameter for which  $S_{\phi_0} = 1$ . In the local Langlands correspondence for  $\mathcal{G}$ , the enhanced parameter  $(\phi_0, 1)$  corresponds to the Steinberg representation  $\mathrm{St}$  of  $\mathrm{SL}_2(K)$ . Note also that, when  $\phi = \phi(1)$ , we have  $S_\phi = 1$ . We send

$$(1, 1) \mapsto \pi(1), \quad (1, \epsilon) \mapsto \mathrm{St}$$

via the left slanted arrow and

$$(1, 1) \mapsto (\phi(1), 1), \quad (1, \epsilon) \mapsto (\phi_0, 1)$$



via the right slanted arrow. This establishes that the geometric conjecture in [ABPS] is valid for  $\mathbf{Irr}(\mathcal{G})^\mathfrak{s}$ .  $\square$

Let  $L/K$  be a quadratic extension of  $K$ . Let  $\lambda$  be the quadratic character which is trivial on  $N_{L/K}L^\times$ . Then  $\lambda$  factors through  $\mathrm{Gal}(L/K) \simeq K^\times/N_{L/K}L^\times \simeq \mathbb{Z}/2\mathbb{Z}$  and  $\phi(\lambda)$  factors through  $\mathrm{Gal}(L/K) \times \mathrm{SL}_2(\mathbb{C})$ . The parameters  $\phi(\lambda)$  serve as parameters for the  $L$ -packets in the principal series of  $\mathrm{SL}_2(K)$ .

It follows from §3 that, when  $K = \mathbb{F}_q((\varpi))$ , there are countably many  $L$ -packets in the principal series of  $\mathrm{SL}_2(K)$ .

**7.1. The tempered dual.** If we insist, in the definition of  $T^\mathfrak{s}$ , that the unramified character  $\psi$  shall be unitary, then we obtain a copy  $\mathbb{T}^\mathfrak{s}$  of the circle  $\mathbb{T}$ . We then obtain a compact version of the commutative triangle, in which the tempered dual  $\mathbf{Irr}^{\mathrm{temp}}(\mathcal{G})^\mathfrak{s}$  determined by  $\mathfrak{s}$  occurs on the left, and the bounded enhanced parameters  $\mathfrak{L}^b(G)^\mathfrak{s}$  determined by  $\mathfrak{s}$  occur on the right. We now isolate the bijective map

$$(21) \quad (\mathbb{T}^\mathfrak{s} // W^\mathfrak{s})_2 \rightarrow \mathbf{Irr}^{\mathrm{temp}}(\mathcal{G})^\mathfrak{s}$$

and restrict ourselves to the case where  $\mathbb{T}^\mathfrak{s}$  contains two *ramified* quadratic characters. Let  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ ,  $W := \mathbb{Z}/2\mathbb{Z}$ . We then have  $T^\mathfrak{s} = \mathbb{T}$ ,  $W^\mathfrak{s} = W$  and the generator of  $W$  acts on  $\mathbb{T}$  sending  $z$  to  $z^{-1}$ .

The left-hand-side and the right-hand-side of the map (21) each has its own natural topology, as we proceed to explain.

The topology on  $(\mathbb{T} // W)_2$  comes about as follows. Let

$$\mathbf{Prim}(C(\mathbb{T}) \rtimes W)$$

denote the primitive ideal space of the noncommutative  $C^*$ -algebra  $C(\mathbb{T}) \rtimes W$ . By the classical Mackey theory for semidirect products, we have a canonical bijection

$$(22) \quad \mathbf{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T} // W)_2.$$

The primitive ideal space on the left-hand side of (22) admits the Jacobson topology. So the right-hand side of (22) acquires, by transport of structure, a compact non-Hausdorff topology. The following picture is intended to portray this topology.



The reduced  $C^*$ -algebra of  $\mathcal{G}$  is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of  $\mathcal{G}$ . Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of  $\mathcal{G}$ , see [Dix, 3.1.1, 4.4.1, 18.3.2]. This makes  $\mathbf{Irr}^{\mathrm{temp}}(\mathcal{G})^\mathfrak{s}$  into a compact space, in the induced topology.

We conjecture that these two topologies make (21) into a homeomorphism. This is a strengthening of the geometric conjecture [ABPS]. In that case, the double-points in the picture arise precisely when the corresponding (parabolically) induced representation has two irreducible constituents. This conjecture is true for  $\mathrm{SL}_2(\mathbb{Q}_p)$  with  $p > 2$ , see [P, Lemma 1]. While in conjectural mode, we mention the following point: the standard Borel subgroup of  $\mathrm{SL}_2(K)$  admits countably many ramified quadratic characters and so, following the construction in [ChP], the geometric conjecture predicts that tetrahedra of reducibility will occur countably many times; however, the

$R$ -group machinery is not, to our knowledge, available in positive characteristic, so this remains conjectural.

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